Perturbative approach to the two-dimensional quantum gravity

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Abstract

The main part of this presentation is a review of the original works [14, 15, 16, 17, 18, 19] on the perturbative approach to the 2-dimensional quantum gravity. We discuss the renormalization of the two-dimensional dilaton gravity in a covariant gauge, the form of the quantum corrections to the classical potential, and the conditions of Weyl invariance in a theory of string coupled to $2d$ quantum gravity.

1. Introduction

Recent progress in a nonperturbative formulations of quantum field theory leads to the increasing interest in this field. However at the moment most of the achievements concerns some special simple models which are essentially different from the ones which can be applied to describe the phenomenology. On the countrary any phenomenological models are rather complicated and therefore it is not clear whether it is possible to obtain any rigour nonperturbative information about them. Thus the standard perturbative way of study is still relevant because in a lot of cases it provides us by the results. In such a situation it is quite natural to apply the standard perturbative methods to the more simple models and thus to get deeper understanding of their internal distinguished properties.

For example here we present some results of the perturbative study of the two-dimensional quantum dilaton gravity. Some versions of the theory are linked with the effective induced action of string on the curved wourld sheet [1], they are exactly soluble [2, 3] and therefore it is natural to address the following questions:

i) Is it possible to distinquish these versions perturbatively?

ii) Are there some other models with the similar properties?

iii) What is the peculiarity of the two-dimensional gravity which provides it's simple nature?

iv) Since the two-dimensional gravity is closely related to the theory of the noncritical string, is it possible to obtain some extra information about the last in the framework of such a study?

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Here we shall try to answer these questions. In fact some of them look a bit naive, especially the third one. It is well known that the two-dimensional metric has only one (conformal, for instance) degree of freedom, and it is just the main feature of the two-dimensional gravity. However the problem is not so simple because the standard perturbative approach supposes the general covariance to be preserved at quantum level. In a covariant gauge the two-dimensional metric has three degrees of freedom that is compensated by the FP ghosts. Thus it is very interesting to understand how the special feature of the two-dimensional gravity looks in this gauge. As it will be shown below the main property of the two-dimensional gravity manifest itself in the form of some explicit identity. This identity concerns the components of covariant quantum metric and leads to some properties related with the especially strong dependence on the gauge fixing parameters.

The starting point of our discussion is the theory of D copies of the free massless scalar fields coupled to the 2-dimensional metric. As usual, we suppose that the integration over the string coordinates and the reparametrization ghosts is performed before the integration over the conformal factor [1]. Vacuum guantum effects and the contribution of the reparametrization ghosts lead to the conformal anomaly and the VEV of the trace of the energy-momentum tensor becomes [4].

$$T =< T_\mu{}^\mu >= \frac{1}{16\pi}\, aR, \qquad\qquad a = \frac{D-26}{6} \tag{1}$$

The anomaly which appears in the noncritical dimension of the target space, leads to the equation for the effective action

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\Gamma}{\delta g_{\mu\nu}} = T \tag{2}$$

which can be solved in the following nonlocal form [1].

$$\Gamma[g_{\mu\nu}] = \frac{1}{16\pi}\, a \int d^2x\sqrt{-g_x} \int d^2y\sqrt{-g_y}\, R(x) \left(\frac{1}{\Box}\right)_{x,y} R(y) \tag{3}$$

The last action can be rewritten in the local form with the help of the dimensionless auxiliary scalar Φ.

$$S_g = \int d^2\sigma\sqrt{g}\, \left\{\frac{1}{2}\, g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + C_1 R\Phi + V(\Phi)\right\} \tag{4}$$

with

$$C_1 = \frac{1}{2}\left(\frac{a-1/6}{2\pi}\right)^{1/2}, \qquad\qquad V(\Phi) = 0 \tag{5}$$

Such an action describes the induced $2d$ gravity for the noncritical string theory, and therefore one must take into account it's quantum effects. Indeed one can start with the generalized theory and regard C_1 as an arbitrary constant and $V(\Phi)$ as an arbitrary function of the field Φ. This is just that we shall do. Below we discuss in details the renormalization of the theory (4) in an arbitrary covariant gauge as well as in the conformal gauge. Then we consider the effective potential and establish the gauge dependence of quantum corrections. The methods of calculations developed here turns out to be very useful and enables one to consider the integration over the string coordinates and the two-dimensional metric without special order. In this case the action of the sigma model in a background fields can be regarded as the direct generalization of (4). The use of covariant gauge enables us to explore the role of the special order of integrations in a consistent manner.

There were a lot of interesting and important investigations in the field of 2 and $2+\varepsilon$ dimensional quantum gravity (see, for example, [5, 6, 7, 8, 9, 2, 3, 8, 9, 10, 11, 12]). The perturbative approach has been developed in [13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. Here we review and discuss the results of [14, 15, 16, 17, 18, 19].

[1] Another order of integrations will be discussed in the consequent sections.

2. The action of induced gravity and it's transformation properties

As it was pointed out in [14, 20] the different forms of the action for the two - dimensional dilaton quantum gravity are equivalent to the expression (4). Since this fact has direct relation to our study let us consider it in some details. One can introduce another scalar φ and also perform the conformal transformation of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\, e^{2\sigma(\varphi)} \tag{6}$$

In terms of new variables the starting action (4) becomes

$$S_n = \int d^2\sigma\sqrt{\bar{g}}\, \left\{\frac{1}{2}\bar{g}^{\mu\nu}\left[\Phi'^2 + 4C_1\Phi'\sigma'\right]\partial_\mu\Phi\partial_\nu\Phi + C_1\bar{R}\Phi + e^{2\sigma}V(\Phi)\right\} \tag{7}$$

where $\Phi = \Phi(\varphi)$, $\sigma = \sigma(\varphi)$ and the primes denote the derivatives with respect to φ. Thus it is clear that the actions

$$S = \int d^2\sigma\sqrt{g}\, \left\{\frac{1}{2}g^{\mu\nu}g(\Phi)\partial_\mu\Phi\partial_\nu\Phi + c(\Phi)R + U(\Phi)\right\} \tag{8}$$

with an arbitrary functions $g(\Phi)$ and $c(\Phi)$ are linked by the conformal transformation of the metric supplemented by the reparametrization of the dilaton, and the corrresponding change of the potential function. In particular one can consider the theory

$$S_{g'} = \int d^2\sigma\sqrt{g}\, \left\{C_1R\phi + v(\phi)\right\} \tag{9}$$

which is classically equivalent to the original theory (4) (with accuracy to some change of the potential, that will be established below).

One can easily see that the above transformation has direct relation to the renormalization of the theory. If we work in the covariant formalism, then all the possible divergences are local covariant expressions. Taking into account the power counting it is easy to prove that the possible divergences in the theory (4) have the structure similar to (8).

$$\Gamma_{div} = \int d^2\sigma\sqrt{g}\, \left\{\frac{1}{2}g^{\mu\nu}A_1(\Phi)\partial_\mu\Phi\partial_\nu\Phi + A_2(\Phi)R + A_3(\Phi)\right\} \tag{10}$$

Since (4) and (8) are conformally equivalent, the first one is renormalizable. Moreover one can establish the renormalzation of the the "more general" theory (8) with the help of divergences calculated in a "particular case" (4). For instance, one can start with (8), transform it to (4) with different field variables and than perform an inverse transformation of the variables in an expression for the divergences. Indeed this inverse transformation can be affected by quantum corrections. However at one loop order it is possible to use the naive classical invertion. [2]

3. The one-loop calculations

To perform the one-loop calculations we apply the background field method and the standard Schwinger-DeWitt technique of extracting the divergences. The special features of the theory requires the special choice of the gauge fixing condition which enables us to investigate the spectrum of the differential operators of the nonstandard form.

Let's make the splitting of the fields $g_{\mu\nu}$ and Φ into the background and quantum parts.

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2}hg_{\mu\nu}, \quad \bar{h}_{\mu\nu}g^{\mu\nu} = 0$$

[2] We remark that it is not obviously true for the dimensions higher than two.

$$\Phi \to \Phi' = \Phi + \phi \tag{11}$$

Here $g_{\mu\nu}$ and Φ are background fields, the $\bar{h}_{\mu\nu}$ is the traceless part of the quantum metric. We divide the quantum metric into trace h and traceless part for the sake of convenience. The one - loop effective action is defined by the bilinear (with respect to the quantum fields) part of the classical action. Since the corresponding bilinear form is degenerate, we have to introduce the gauge fixing term and the action of ghosts. The more general form of the covariant gauge fixing term is following:

$$S_{gf} = -\int d^2\sigma\sqrt{g}\;\chi_\mu\; G^{\mu\nu}\;\chi_\nu \tag{12}$$

where χ_μ is the background gauge and $G^{\mu\nu}$ is the weight operator.

$$\chi_\mu = \nabla_\nu h^\nu_\mu - \beta\nabla_\mu h - \gamma\nabla_\mu\phi - E^{\rho\sigma}_\mu h_{\rho\sigma} - F_\mu\phi, \qquad G^{\mu\nu} = \tau g^{\mu\nu} \tag{13}$$

Here $\tau,\ \beta,\ \gamma,\ E^{\rho\sigma}_\mu,\ F_\mu$ are arbitrary functions (gauge parameters) of the dimensionless background field Φ. Let us now make some remark conserning the choice of these functions. For the sake of simplicity we require the bilinear form of the total action $S^{(2)} + S_{gf}$ to be minimal. The last means that the second derivatives of the quantum fields appear only in the combination $\Box = \nabla_\nu\nabla^\nu$. In the four dimensional gravity such a condition fixes all the gauge parameters (see, for example, [29]). On the countrary, in the case of the two-dimensional gravity we can apply the identity proved in [19] (see also another proof in [16]) for the particular case of the flat background metric)

$$\bar{h}^{\mu\nu}\; X\; [\; \frac{1}{2}\delta_{\mu\nu,\alpha\beta}\Box - g_{\nu\beta}\nabla_\mu\nabla_\alpha - \frac{1}{2}\delta_{\mu\nu,\alpha\beta}R\;]\;\bar{h}^{\alpha\beta} = 0 \tag{14}$$

Here X is an arbitrary scalar function. It turns out that (14) allows us to get the minimal operator for an arbitrary value of some gauge parameter ν. At the same time the gauge parameters $\tau,\ \beta$ and γ_a must be taken in a special way to provide the minimality of the total action.

$$\beta = 0, \qquad \gamma = -\frac{\nu}{\Phi}, \qquad \tau = \frac{\Phi}{\nu} \tag{15}$$

Furthermore there remains an arbitrariness related with the functions $E^{\rho\sigma}_\mu$ and F_μ. The explicit calculation shows that the divergent part of the one - loop effective action does not depend on these functions [14]. In fact the lack of such dependence can be proved in a general form, that will be demonstrated below.

For the special "minimal" choice of the gauge fixing the bilinear part of the total action $S + S_{gf}$ has the form

$$(S + S_{gf})^{(2)} = \frac{1}{2}\int d^2\sigma\sqrt{g}\;(\bar{h}_{\mu\nu},\ h,\ \ \phi)(\hat{H})(\bar{h}_{\alpha\beta},\ h,\ \ \phi)^T \tag{16}$$

where T denotes transposition, and the self - adjoint operator $\hat{H}$ has the structure

$$\hat{H} = \hat{K}\Box + \hat{L}^\lambda\mathcal{D}_\lambda + \hat{M} \tag{17}$$

Here $\hat{K}, \hat{L}^\lambda, \hat{M}$ are c-number operators acting in the space of the fields $(\bar{h}_{\rho\sigma},\ h,\ \phi)$ and $det||\hat{K}||$ is nonzero. One can find an explicit expressions in [15].

The action of the Faddeev-Popov ghosts is defined in a usual way and has the form [15].

$$S_{gh} = \int d^2\sigma\sqrt{g}\;\bar{C}^\alpha\; M^\beta_\alpha\; C_\beta \tag{18}$$

The one - loop divergencies of the effective action are given by the expression

$$\Gamma^{(1)}_{div} = -\frac{1}{2}\;Trln\hat{H}\Big|_{div} + Trln\hat{M}_{gh}\Big|_{div} \tag{19}$$

The form of the operator $\hat{M}_{gh}$ enables us to apply the standard Schwinger-DeWitt technique of the divergences calculation. On the other hand, the structure of the operator $\hat{H}$ (17) enables us perform the following transformation.

$$Trln\hat{H} = Trln\hat{K} + Trln(\hat{1}\Box + \hat{K}^{-1}\hat{L}^{\lambda}\nabla_{\lambda} + \hat{K}^{-1}\hat{M}) \tag{20}$$

First term in (20) does not give contribution to the divergences, because $\hat{K}$ is local operator and therefore this term can be omitted. The second term has standard structure as well as the ghost action operator. Now it is possible to derive the one-loop divergences, applying the well-known result of the Schwinger-De Witt expansion. The resulting values of A_1, A_2, A_3 are [15, 19].

$$A_1 = -\frac{\nu}{\varepsilon\Phi^2}, \qquad A_2 = \frac{1}{6\varepsilon}\, R, \qquad A_3 = \frac{1}{\varepsilon}\left[\frac{\nu V}{C_1\Phi} + V'\frac{1}{C_1}\right] \tag{21}$$

where $e = 2\pi(d-2)$ is the parameter of dimensional regularization. Let us make some comments on the above result.

i) The gauge dependence of the divergences is proportional to the dynamical equation $g_{\mu\nu}\,\frac{\delta S_g}{\delta g_{\mu\nu}}$. Therefore it can be removed by the renormalization of the metric. The remaining divergences are in the potential A_3 sector and also the topological Einstein divergences in A_2 sector. On shell the one-loop divergences are gauge independent as it has to be.

ii) From the previous point it follows that the divergences in the kinetic sector are essentially determined by the ones in the potential sector. Since the gauge fixing parameters E, F (13) may affect only the kinetic type divergences and therefore any dependence on these parameters is forbidden. We note that this follows from general backgrounds and also agrees with the result of direct calculations.

iii) In the conformal gauge the theory (4) becomes the linear $D = 2$ sigma model with the nontrivial tachyon sector only [20]. It is well known that in such a model only the tachyon type divergences may arise, and therefore the above result can be regarded as the confirmation of this known fact. Indeed we have used the covariant gauge where (4) is not equivalent to the ordinary sigma model. In this sence our calculation gives more general information.

iv) The coefficient of the pure gravitational diveregence A_2 also corresponds to the $D = 2$ sigma model. It is remarcable that it can be calculated in a direct way in a harmonic gauge, without the study of the reparametrization ghosts *etc* as it have been done in [1]. The calculation of this counterterm is essentially based on the identity (14) and shows that this coefficient does not depend on the gauge parameter ν. One can see that the one-loop divergences of the Einstein type are uniquely defined in covariant gauge. Thus one can consider the renormalization of the composite operator $< T^{\mu}_{\mu} >$ and obtain the expression for the conformal anomaly in a covariant gauge. We remark that the R-type counterterm is defined in a unique way, and therefore the corresponding trace anomaly is defined in a unique way in the harmonic gauge. In this sence our result exactly corresponds to the calculation of the anomaly performed earlier in a harmonic gauge in [23, 24, 25, 26, 27] and also discussed in [28]. Indeed the calculation of the counterterm is essentially more simple if we take into account the identity (14). Below a more general calculation of $< T^{\mu}_{\mu} >$ will be considered in some details.

At the end of this section we discuss the one-loop divergences of the more general dilaton theory (8). According to our previous consideration one can extract these divergences transforming (21). It is possible to verify this fact with the help of some simple particular case. To do this let us consider the conformal transformation between the model (4) and the model without kinetik term for the dilaton (9). Direct calculation performed in the model (9) shows that in the minimal gauge the divergences has exactly the same form (21). This makes the analysis fairly easy. The conformal transformation, reparametrization of the dilaton and the transformation of the potential functions

which link (4) and (9) are [for the sake of convenience we denote the metric of the model (4) as $\bar{g}_{\mu\nu}$, see (6)].

$$\Phi = \phi + \phi_0, \qquad \phi_0 = const.$$

$$\sigma = \frac{1}{4C_1}\,\phi + \sigma_0, \qquad \sigma_0 = const.$$

$$v(\phi) = V(\Phi) exp\left(\frac{1}{2C_1}\Phi + 2\sigma_0\right) \tag{22}$$

It is useful to put $\sigma_0 = \phi_0 = 0$. The inverse transformation corresponds to the conformal factor $-\sigma$ and to $V = ve^{-2\sigma}$.

Now we have two ways to derive the one-loop divergences in the theory (9) in a harmonic gauge. The first one is the direct calculation similar to that we have just presented for the theory (4). The second way is to change variables according to (22), then use (21) and then make an inverse transformation. After some calculations we arrive at the following results. As it was already mentioned, on the first way we obtain (21). The calculation in transformed field variables gives different result in the potential sector

$$\bar{A}_3 = \frac{1}{\varepsilon}\left[\frac{\nu V}{C_1 \Phi} + V' \frac{1}{C_1} - \frac{V}{2{C_1}^2}\right] \tag{23}$$

The origin of this difference is the different choice of the field quantum variables. One can easily check that on the classical equations of motion the difference in two expressions concerns only the surface counterterm $\Box\Phi$. And so we have seen that our consideration of the renormalization of the theory (8) was true. One can establish the corresponding counterterms using the result (21) computed for the model (4) and the formula (6).

4. The effective potential and gauge dependence

It is a well know fact that in the gauge theories the effective potential depends on the gauge fixing parameters. In the two dimensional dilaton gravity such a dependence is even stronger than in other cases. We define the effective potential $V_{eff}(\Phi)$ as the part of effective action which survives on the constant background $\Phi = const, R = 0$. So A_3 in (21) is the divergent part of $V_{eff}(\Phi)$. One-loop effective action is given by the expression

$$V_{eff}^{(1-loop)} = V - \Delta V - \frac{1}{2} Tr \sum_{k=1}^{4} ln\lambda_k + Tr \sum_{l=1}^{2} ln{\lambda'}_l \tag{24}$$

where λ_k and ${\lambda'}_l$ are the eigenvalues of the operators $\hat{H}$ and $\hat{M}_{gh}$ respectively. Tr includes the integration over the momentums in the framework of some regularization scheme. (24) together with the identity (14) give the possibility to derive the one-loop effective potential for arbitrary gauge fixing parameters ν, β, γ. One can find an explicit most general expression for V_{eff} in [16] where the cut-off regularization have been used. Just as in the conformal gauge [20] the logarithmical divergences of the effective potential V_{eff} depend on V'' as well as the finite part, but only for the nonminimal gauges.

However the most interesting is the minimal gauge $\beta = 0$, $\gamma = \nu\Phi^{-1}$. As it was already noted above, the identity (14) leads to that the "natural" configuration space metric in the theory (4) depends on the gauge parameter ν.

Taking the counterterm in the form

$$\Delta V = +\frac{1}{4\pi}\mu^2 A(\Phi) ln\left(\frac{\Lambda^2}{\mu_2}\right) + \frac{1}{4\pi}\Lambda^2 B(\Phi), \tag{25}$$

where Λ is the cut-off regularization parameter, μ^2 is the dimensional parameter of renormalisation, and A, B are some unknown functions, we require $V_{eff}^{(1-loop)}$ to be finite and after determination of A and B we finally get [16]

$$V_{eff}^{(1-loop)} = V - \frac{1}{4\pi}\left\{\frac{V'}{C_1}\left[1 - ln\left(\frac{V'}{C_1\mu^2}\right)\right] + \frac{\alpha V}{C_1\Phi}\left[1 - ln\left(\frac{\alpha V}{C_1\Phi\mu^2}\right)\right]\right\} \tag{26}$$

The expression (26) has to be supplemented by the normalization conditions. Let us, for example, consider the exactly soluble case $V(\Phi) = \Omega\Phi$. Then the quantum correction in (26) is some constant which doesn't depend on Φ. Introducing the condition

$$V_{eff}^{(1-loop)}(\Phi = 0) = 0 \tag{27}$$

we find $V_{eff}^{(1-loop)} = V$ that is the absence of all the quantum corrections. For the other interesting case $V_{eff}^{(1-loop)} = \Omega exp(\lambda\Phi)$ we take the normalization condition of the form

$$V_{eff}^{(1-loop)}(\Phi = 0) = \Omega \tag{28}$$

and find that quantum corrections do not vanish even for the value $\alpha = 0$, and do not repeat the structure of V.

Since we are interested here in the general form of the potential function which provides the triviality of the one-loop corrections, the gauge dependence of this quantity leads to some problem. Since V_{eff} is gauge dependent it is not clear what means the correct choice of gauge. Thus it is natural to try to use the unique effective action of Vilkovisky [30] which is supposed to give the gauge-independent result. At the same time there is the well - known problem in the unique effective action which contains the dependence on the metric in the space of fields (configuration space) [31, 32, 33]. The additional condition of Vilkovisky [30] fixes this metric to be natural (that is the matrix in the bilinear form of the action in a minimal gauge). This condition implies the essential use of the classical action of the theory. However in the theory (4) even the natural configuration space metric depends on the gauge parameter ν 2The problem with a choice of the "natural" metric in $d = 2$ gravity has been pointed out already in the original work of Vilkoviski [30]. and therefore such dependence may arise in the unique effective potential.

The direct calculations have shown, that the gauge dependence of the Vilkovisky effective potential really takes place [16]. Note that it is also follows from the calculations of Kantowski and Marzban [21] with the natural metric of the form:

$$G_{ij} = \begin{pmatrix} \frac{C_1\Phi}{2\nu} & 0 & 0 \\ 0 & 0 & -C_1/2 \\ 0 & -\frac{C_1}{2} & (1 - \frac{C_1\nu}{\Phi}) \end{pmatrix} \tag{29}$$

that corresponds to the matrix $\hat{K}$ in (17). Thus the gauge dependence of the effective potential in the theory (4) is even stronger than usually, in the sence that it can not be removed by taking into account the Vilkoviski corrections. Thus the only one thing we can do is to choose some special "correct" gauge. In view of the special form of the gauge dependence this corresponds to some special conformal reparametrization of the background metric.

Let us now turn to the search of the theories with the trivial quantum corrections to $V(\Phi)$. We shall take the condition of triviality in the form:

$$V_{eff}^{(1-loop)} = (1 + \tau)V + \eta \tag{30}$$

where τ, η are some constants. If we choose the gauge with the vanishing kinetic divergent contributions to the effective action then there are only three appropriate gauges: i) light - cone,

ii)conformal and iii)harmonical gauge with $\nu = 0$. Since in the last case the loop corrections to the effective potential look more simple let us consider iii). Then (30) is rewritten in the form

$$\tau V + \eta = \frac{V'}{4\pi C_1} ln \left(\frac{V'}{eC_1\mu^2} \right) \tag{31}$$

This is an ordinary differential equation which can be solved easily. In the case of $\tau = 1$ there is only one (well - known) solution $V(\Phi) = \Omega\Phi + \Omega_1$. where Ω, Ω_1 are some constants. However if τ is not equal to 1 there are two additional solutions of the form:

$$V = -\frac{\eta}{\tau} + \frac{\mu^2}{4\pi\tau} \left[-1 \pm \sqrt{8\pi C_1 \tau (\Phi - \Phi^*)} \right] e^{\pm\sqrt{8\pi C_1 \tau (\Phi - \Phi^*)}} \tag{32}$$

Here Φ^* is the constant of integration. The expression (32) contains an arbitrary parameters η, τ and moreover the extra dependence on the renormalization parameter μ. The last dependence has to be removed by the introduction of some normalization conditions. If one uses the condition (30) then the nontrivial part of the renormalized effective potential turns out to be zero. On the other hand it is much more natural to take the more general normalization condition of the form

$$V_{eff}^{(1-loop)}(0) = \tau^* V(0) + \eta^* \tag{33}$$

where τ^*, η^* are some constants (which may coincide with τ, η). Then it is possible to solve (33) and to express μ^2 in terms of $\tau^*, \eta^*, \tau, \eta$.

5. Covariant calculation for string coupled to quantum gravity

In standard string theory it is accepted to consider the path integral in two steps [1]. First the integration over string coordinates is fulfilled and then the one over the two-dimensional metric $g_{\mu\nu}$. After the first integration the condition of Weyl invariance is introduced and after that the intergal over the two-dimensional metric is reduced to the integral over only modulus. The conditions of Weyl invariance produce the effective equations for the background fields, these equations appears as a series in α' [34, 35, 36, 37, 38, 39]. In what follows we shall name this approach as the standard one.

It is usually assumed that the result does not depend on the order of integrations (see, for example, the papers [40, 41]). In both papers the conformal gauge has been used. In this gauge the D dimensional sigma model coupled to quantum gravity can be transformed to the $D+1$ dimensional sigma model on classical gravitation background whereas the conformal factor of the metric plays the role of an extra coordinate. The qualitative analysis of the quantum gravity effects leads to conclusion that the Weyl invariance of the theory puts additional restrictions on the starting D dimensional sigma model [40]. More detailed analysis including the qualitative account of contribution of the Jacobian of the conformal transformation shows that all the effects of quantum gravity concerns the additional $D+1$ dimensional couplings which have to be introduced for the sake of renormalizability. As a result the effective equations in original D dimensions are the same as in the case when gravity is classical background [41].

All the conclusions of [40, 41] are based on the use of conformal gauge. In this gauge the renormalizability may be broken by the terms, depending on conformal factor. Therefore it is quite reasonable to explore the problem in a covariant gauge where the renormalization structure is fairly simple [19]. Our purpose is to investigate whether the quantum gravity leads to any nontrivial changes in the vacuum expectation value of the trace of the Energy - Momentum Tensor $< T^\mu_\mu >$. To do this it is necessary to make explicit calculations in a covariant gauge (or extract the divergences with the help of some anzats) and put $< T^\mu_\mu >= 0$ that gives the one-loop conditions

of Weyl invariance. Even if these conditions are different from the standard effective equations, it doesn't mean, of course, that the last are uncorrect. In fact one has to check, whether the given loop approximation in the theory with quantum metric corresponds to the same approximation in the standard approach. In other words one must check that the α' is the parameter of the loop expansion even if the two dimensional metric and sigma model coordinates are considered on an equal footing. If this point is not satisfied whereas the effective equations are different, we arrive at the conditions of the Weyl invariance which are qualitatively different from the standard ones but do not contradict them. Here we show that it is just the case. It turns out that the one loop conditions for the Weyl invariance of must be essentially modified if the effects of quantum metric are taken into account. However the change of the order of integrations disturbs the standard logical construction, related with the Weyl invariance conditions. If the two dimensional metric is quantized then α' is no longer the parameter of the loop expansion. Thus there is no small parameter which can keep under control the contributions of higher loops, and therefore in this case the "effective equations" have restricted sence. For instance, one can construct such "effective equations" in nth loop but these equations will not preserve the part corresponding to the $n-1$ loop. In spite of this it is believed that the study of the string theory interacted with $2d$ gravity can be useful for the better understanding of the general link between quantum gravity and the theory of strings.

If we deal with the one-loop approximation only, then the difference between the divergences calculated in two different gauges must be proportional to the classical equations of motion. Therefore one can apply more comfortable conformal gauge to derive the conditions of Weyl invariance and then verify the equivalence with the covariant gauge. On this way we can compare the qualitative consideration of Ref.'s [40, 41] with the result of direct calculation in the covariant gauge.

The action of closed boson string in the massless and tachyon background fields has the form:

$$S = \int d^2\sigma\sqrt{g}\ \{\frac{1}{2\alpha'}g^{\mu\nu}G_{ij}(X)\partial_\mu X^i\partial_\nu X^j + \frac{1}{\alpha'}\frac{\varepsilon^{\mu\nu}}{\sqrt{g}}A_{ij}(X)\partial_\mu X^i\partial_\nu X^j + B(X)R + T(X)\} \tag{34}$$

Here $i, j = 1, 2, ..., D$; $\mu, \nu = 1, 2$; G_{ij} is the background metric, A_{ij} is the antisymmetric tensor background field, $B(X)$ is the dilaton field, $T(X)$ is the tachyon background field. R is the two dimensional curvature, $X^i(\sigma)$ are the string coordinates. At quantum level, since we do not follow the order of integrations, the two dimensional metric is quantum field. That is why one can regard (34) as the action for the two dimensional quantum gravity.

It is easy to see that two actions (34) and (4) have a very similar structure and hence one can start directly with the action (34), regarding $g_{\mu\nu}$ as quantum field. In fact the results (at the one-loop level) are qualitatively the same even if the sum of (34) and (4) is choosen as the starting action [18]. Let us notice that the theory with the action $S + S_g$ is equivalent to the $D+1$ dimensional sigma model

$$S = \int d^2\sigma\sqrt{g}\ \{\frac{1}{2\alpha'}g^{\mu\nu}G_{ab}(Y)\partial_\mu Y^a\partial_\nu Y^b + \frac{1}{\alpha'}\frac{\varepsilon^{\mu\nu}}{\sqrt{g}}\ A_{ab}(Y)\partial_\mu Y^a\partial_\nu Y^b + \mathcal{B}(Y)R + T(Y)\} \tag{35}$$

with the special restrictions on the background fields [40, 41, 17, 18]. Here the indeces $a, b, ..$ take the values $1, 2, ..., D+1$. Below we deal only with the model (35), omitting the mentioned restrictions for the sake of brevity.

As far as we consider only the one-loop renormalization, the gauge dependence of divergences have to be proportional to the classical equations of motion. For the sake of simplicity we write only some (most relevant, because all the information is preserved) combination of these equations, which follow from the action (35).

$$E_g = T - \Box\mathcal{B} = 0$$

$$E_Y = -T + \mathcal{B}_{ab} g^{\mu\nu} \partial_\mu Y^a \partial_\nu Y^b + \mathcal{B}_a H^a_{bc} \frac{\varepsilon^{\mu\nu}}{\sqrt{g}} \partial_\mu Y^b \partial_\nu Y^c + \alpha' \mathcal{B}_a \mathcal{B}^a R + \alpha' \mathcal{B}_a T^a = 0 \tag{36}$$

The indices a, b, c near $\mathcal{B}$, T and Y indicate to the covariant derivatives in target $D+1$ dimensional space. H and K (below) are the torsion and corvature tensors based on G and A.

The renormalizability of the theory (35) follows from the power counting consideration. Indeed it is important that it is possible to preserve the diffeomorphism invariance on the wourld sheet and also reparametrization invariance in target space on quantum level. It is fairly easy to see that we are able to preserve both symmetries, because there exists the covariant (i.e. dimensional) regularization. It is also relevant that we can work in harmonic type (covariant) gauge which is not sensitive to this regularization. However for our purposes it is very convenient to apply the conformal gauge in parallel.

If one introduces the conformal background gauge, then the action (35) can be expressed in the form of the $D+2$ dimensional nonlinear sigma-model of the special form [40, 41]. The two dimensional metric in this case is not quantized. The conformal background gauge is introduced by the relation

$$g_{\mu\nu} = e^{-2\rho} \bar{g}_{\mu\nu} \tag{37}$$

where $\bar{g}_{\mu\nu}$ is some fixed background metric. In this gauge the action (35) becomes

$$S = \int d^2\sigma \sqrt{\bar{g}} \left\{ \frac{1}{2\alpha'} \bar{g}^{\mu\nu} G_{AB}(Z) \partial_\mu Z^A \partial_\nu Z^B + \frac{\varepsilon^{\mu\nu}}{\alpha' \sqrt{\bar{g}}} A_{AB}(Z) \partial_\mu Z^A \partial_\nu Z^B + \mathcal{B} R(\bar{g}) + T \right\} \tag{38}$$

where $A, B = 1, 2, ..., D+2$ and

$$Z^A = (\rho,\ Y^a), \qquad G_{AB} = \begin{pmatrix} 0 & -2\alpha' \mathcal{B}_b \\ -2\alpha' \mathcal{B}_b & G_{ab} \end{pmatrix}, \qquad A_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & A_{ab} \end{pmatrix} \tag{39}$$

Below we shall use both (35) and (38) forms of the action.

The action (35) is invariant under the general coordinate transformations in two dimensional and also in $D+1$ dimensional spaces. However, since the two dimensional metric is quantized, it is natural to expect that some extra symmetry takes place. Such symmetry turns out to be the particular form of the reparametrizations of the $D+2$ dimensional model (38), which preserves the block structure of the background metric (39) [19]. The generalised symmetry cause an arbitrariness of renormalization, so the last is related to the reparametrizations and also to the conformal transformation of the two-dimensional metric.

The calculation of divergent part of the one loop effective action in the theory (35) is performed in a way similar to the one we have used above for the theory (4). The difference is that here we apply the background field method simultaneously in two dimensional and $D+1$ dimensional covariant form.

The background shift of the fields $g_{\mu\nu}, Y^a$ is performed according to (11) and

$$Y^a \to Y'^a = Y^a + \sqrt{\alpha'}\, \pi^a(\eta^a) \tag{40}$$

The expansion (40) supposes the standard use of the normal coordinate method. The quantum field π^a is parametrized by the tangent vector η^a in $D+1$ dimensional target space (see [18, 19] for the notations).

The more general form of the gauge fixing term, which corresponds to the covariant harmonic gauge is following:

$$S_{gf} = -\int d^2\sigma \sqrt{g}\, \chi_\mu\, G^{\mu\nu}\, \chi_\nu \tag{41}$$

where χ_μ is the background gauge and $G^{\mu\nu}$ is the weight operator.

$$\chi_\mu = \nabla_\nu h^\nu_\mu - \beta \nabla_\mu h - \gamma_a \nabla_\mu Y^a - E^{\rho\sigma}_\mu h_{\rho\sigma} - F_{\mu a} Y^a, \qquad G^{\mu\nu} = \tau g^{\mu\nu} \tag{42}$$

Here $\tau,\ \beta,\ \gamma_a,\ E^{\rho\sigma}_{\mu},\ F_{\mu a}$ are arbitrary functions (gauge parameters). which are taken in a special way to provide the minimality of the total action.

$$\beta = 0, \qquad \gamma_a = -\frac{\nu \mathcal{B}_a}{\mathcal{B}}, \qquad \tau = \frac{\mathcal{B}}{\nu} \tag{43}$$

There remains an arbitrariness related with the parameter ν and also the unessential one related with the functions $E^{\rho\sigma}_{\mu}$ and $F_{\mu a}$. The explicit calculation shows that the divergent part of the one - loop effective action does not depend on these functions just as in the pure gravity theory (4).

The bilinear part of the total action $S + S_{gf}$ has the form

$$(S + S_{gf})^{(2)} = \frac{1}{2} \int d^2\sigma \sqrt{g}\, (\bar{h}_{\mu\nu},\ h,\ \ \eta^a)(\hat{H})(\bar{h}_{\alpha\beta},\ h,\ \ \eta^b)^T \tag{44}$$

where the self - adjoint operator $\hat{H}$ has the form (17). Here $\hat{K}, \hat{L}^{\lambda}, \hat{M}$ are c-number operators acting in the space of the fields $(\bar{h}_{\rho\sigma},\ h,\ \eta^a)$. The complete set of technical details are presented in [18]. Here we write down only the matrix $\hat{K}$.

$$\hat{K} = \begin{pmatrix} \frac{\mathcal{B}}{2\nu} P^{\mu\nu,\,\alpha\beta} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\mathcal{B}_b\sqrt{\alpha'} \\ 0 & -\frac{1}{2}\mathcal{B}_a\sqrt{\alpha'} & -G_{ab} + \frac{\nu\alpha'}{\mathcal{B}}\mathcal{B}_a\mathcal{B}_b \end{pmatrix} \tag{45}$$

Here $P^{\mu\nu,\,\alpha\beta}$ is the projector to the traceless states. It is important that the first term in $\bar{H}$ corresponds to the propagator of the theory (35). Since $\hat{K}$ is inhomogeneous in α', this constant does not play as the parameter of the loop expansion in the theory with quantum metric.

The action of the Faddeev-Popov ghosts is defined in a usual way. Summing up the contributions of $Tr \ln \hat{H}$ and the ghost part we get [17]

$$\Gamma^{(1)}_{div} = -\frac{1}{\varepsilon}\int d^2\sigma\sqrt{g}\, \Bigg\{ \left[\frac{24 - D}{12} + \frac{\alpha'}{2}(\mathcal{D}^2\mathcal{B}) - \frac{\alpha'}{2}\frac{\mathcal{B}^a\mathcal{B}_{ab}\mathcal{B}^b}{\mathcal{B}^c\mathcal{B}_c} \right] R \ - \ \frac{\nu}{\mathcal{B}}(\Box\mathcal{B} - T) +$$
$$+ \frac{1}{2}\mathcal{K}^a_a + \frac{\alpha'}{2}\mathcal{D}^2 T + \frac{1}{\mathcal{B}^c\mathcal{B}_c}[\, \mathcal{B}^a T_a - \frac{\alpha'}{2}\mathcal{B}^a T_{ab}\mathcal{B}^b\,] - \frac{1}{2\mathcal{B}^c\mathcal{B}_c}[\, \frac{1}{\alpha'}\Box\mathcal{B} + \frac{1}{2}\mathcal{B}^b\Box\mathcal{B}_b -$$
$$-\frac{1}{2}g^{\mu\nu}\mathcal{B}^a_\mu\mathcal{B}_{a\nu} + \mathcal{B}^a\mathcal{B}^b(g^{\mu\nu}H_{fbc}H^f{}_{ea}\,\partial_\mu Y^e\partial_\nu Y^c + \mathcal{K}_{ab}) - 2\mathcal{B}^b\mathcal{B}_{da}\frac{\varepsilon^{\mu\nu}}{\sqrt{g}}H^d{}_{eb}\partial_\mu Y^a\partial_\nu Y^e] \Bigg\} \tag{46}$$

One can compare the divergent part of the one - loop effective action (46) with the known results of other papers. For example, the divergence of Einstein type in (46) has standard form and does not depend on the gauge parameter ν, that is just the expected result for the anomaly contribution. Next, if one puts $D = 0, \quad \mathcal{B} = C_1\Phi$ and $H_{abc} = 0$ and so reduce the theory to the form (4), then (46) coincide with the expression, which was derived in [15, 16] for pure gravity (4). Note also that the ν - dependent terms in (46) are proportional to the equations of motion (36), and hence disappear on mass shell. One can consider this as some kind of control for the correctness of the calculations.

Next feature of (46) corresponds to the separation of quantum gravity contributions, and it is not so obvious. In fact all the terms in (46), which are related with the contributions of quantum metric, exhibit the dependence on ν or contain the factor of $(\mathcal{B}_a\mathcal{B}^a)^{-1}$. Thus removing all the ν - dependent terms and also the terms which have $\mathcal{B}_a\mathcal{B}^a$ in the denominators, we obtain the well-known result for the $D+1$ dimensional sigma - model (35) without quantum gravity [36]. Recently there were published the papers with calculations in the two dimensional quantum gravity coupled to the linear sigma model [22, 47]. The expression (46) is in a good accord with the results of these papers.

The expression (29) looks rather complicated and the appearance of $\mathcal{B}_a\mathcal{B}^a$ terms in some denominators seems strange, but in fact it is quite natural. It turns out that this form of the divergences corresponds to the relations between the geometric quantities like curvatures based on the metric G_{ab} and on the $D+2$ dimensional metric G_{AB} (39). To understand this, we perform the parallel calculation in the conformal gauge.

6. Conformal gauge and the conditions of Weyl invariance

As it was already shown above, in the conformal gauge the theory (35) is the $D+2$ dimensional nonlinear sigma model on the background of purely classical two dimensional metric (38). We need the explicit expression for the one-loop divergences in conformal gauge, because it gives the possibility to derive the VEV of the Energy-Momentum Tensor in a more simple way. In fact there is no need to make special loop calculations in this case. Since in the conformal gauge we are dealing with the ordinary $D+2$ dimensional sigma model with the background fields G and A of special form (39) it is possible to use the well known result for the one-loop divergences of the ordinary sigma-model. On this way we obtain these divergences expressed in terms of the fields T, $\mathcal{B}$, G_{AB} and A_{AB} (39). Then the expression for the divergences can be rewritten in terms of $D+1$ geometrical quantities. The above consideration essentially uses the universal form of the divergences of quantum field theory in an external field [29].

We have performed an independent calculation in conformal gauge and also applied the standard result of Callan, Friedan, Martinec and Perry [35] together with the reduction formulas for the geometric quantities [18, 19]. The results coincide and give the following difference between two effective actions (the symbol c denotes the use of the conformal gauge).

$$\Gamma^{(1c)}_{div} - \Gamma^{(1)}_{div} = -\frac{1}{\varepsilon}\int d^2\sigma\sqrt{g}\,\{\,-\left[\frac{\nu}{\mathcal{B}} + \frac{1}{2\alpha'\mathcal{B}^a\mathcal{B}_a}\right]E_g \;+\; \frac{1}{2\mathcal{B}^c\mathcal{B}_c}\left[\frac{\mathcal{B}^a\mathcal{B}_{ab}\mathcal{B}^b}{\mathcal{B}^c\mathcal{B}_c} - (\mathcal{D}^2\mathcal{B})\right]E_Y\,\} \tag{47}$$

The expression for $\Gamma^{(1c)}_{div}$ is just the known divergent part of the effective action for the ordinary (that is without quantum gravity) sigma-model in the case of the restricted background interaction fields (39). Thus we see that the denominators in (46) reflect the form of the curvature tensor in $D+2$ dimensional target space with special form of the metric (39) in this space.

The expressions for $\Gamma^{(1c)}_{div}$ and $\Gamma^{(1)}_{div}$ do not coincide, nevertheless both are correct and moreover verify the correctness of each other. To see this one can apply the arguments of Ref. [48]. The general theorem [45, 46] (see also [29]) claims that the difference between two effective actions derived in two different gauges must vanishes on mass shell. In particular, for the one-loop divergences this difference vanishes on the classical equations of motion. Both integrands of $\Gamma^{(1c)}_{div}$ and $\Gamma^{(1)}_{div}$ are local functionals of the same dimension as the equations of motion, and therefore the difference $\Gamma^{(1c)}_{div} - \Gamma^{(1)}_{div}$ must be the linear combination of the equations of motion with the local coefficients, that is just the case (47). One can suppose that $\Gamma^{(1c)}_{div}$ and $\Gamma^{(1)}_{div}$ can be converted into each other after some reparametrization transformation and that the corresponding arbitrariness of the interaction fields have the well - known Killing form.

At one loop all the divergences can be removed by the renormalization of the background fields G_{ab}, A_{ab}, $\mathcal{B}$, T and hence the renormalization of the two-dimensional metric is not necessary. From this point of view the renormalization of the model under consideration is qualitatively the same as for the ordinary sigma model (34). However the expressions for the beta functions are essentially more complicated in our case.

Now we can formulate the conditions of Weyl invariance at the one-loop level. In the framework of the standard approach the theory can be formulated in such a manner that the Weyl invariance is the symmetry of the theory on both classical and quantum levels. To do this, it is necessary to

input the loop order parameter α' into the starting action and thus divide this action into two parts. One of these parts, that is classical action, is Weyl invariant, and another one gives the nonzero contributions to the trace of the Energy-Momentum Tensor $< T^{\mu}_{\mu} >$. These contributions contains the additional α' factor. The one loop corrections also give contributions of the same order, and therefore it is possible to provide the Weyl invariance at the one loop level if the corresponding conditions (effective equations) for the background fields are introduced. This scheme can be extended to higher loops. In fact in a usual approach it is also supposed the integration over the two-dimensional metric, but it is performed after the integration over the sigma-model variables when the conditions of Weyl invariance are already satisfied (see, for example, [39]). Then the integration over the metric is reduced to the summation over topologies, that exactly corresponds to the anomaly free Weyl invariant theory in two dimensions.

In the theory under discussion the two-dimensional geometry possesses classical dynamics at classical level. However, from general point of view, if we want to develop a theory having relation to string, the effective action has to be independent on the scale factor. Hence it is necessary to formulate the conditions of the Weyl invariance on quantum level in the theory with quantum gravity and thus check whether there will be some real difference with the standard approach.

The standard way to derive $< T^{\mu}_{\mu} >$ is to start with the action (35) and calculate the renormalization of the composite operators [37, 38]. However in our case we can apply more simple method. The necessity to renormalize $g_{\mu\nu}$ does not appear in the model (35) in both harmonic and conformal gauges. Hence, at one loop level there is no big difference between two gauges and one can apply anyone of them to formulate the effective equations. At the same time in the conformal gauge the model (35) with quantum gravity is equivalent to the ordinary sigma model in higher dimension $D+2$ (38). As far as we know the effective equations for ordinary sigma model and also the relations between the geometrical quantities in dimensions $D+1$ and $D+2$, it is not difficult to derive the effective equations for the model (35).

Since the introduction of the tachyon term leads to the well known difficulty that is to the lack of equivalence between the different approaches to the string theory (see for example [39]), we omit this term here. Applying the reduction formulas [18, 19] we find the effective equations for G, A and B background fields in the following form.

$$\bar{\beta}^{G}_{ab} = K_{ab} + \frac{1}{\mathcal{B}^2}\,[\,\mathcal{B}_{ac}\mathcal{B}^{c}_{b} - \mathcal{B}^{c}\mathcal{B}^{d}K_{acbd} + (\mathcal{D}^2\mathcal{B})\mathcal{B}_{ab}] +$$
$$+\frac{1}{\mathcal{B}^4}[\frac{1}{4}(\mathcal{B}^2)_{,a}(\mathcal{B}^2)_{,b} - \frac{1}{2}(\mathcal{B}^2)_{,c}\mathcal{B}^{c}\,\mathcal{B}_{ab}\,] - \frac{1}{4}H_{a}{}^{cd}H_{bcd} = 0$$
$$\bar{\beta}^{A}_{ab} = \mathcal{D}_c H^{c}{}_{ab} + \frac{1}{\mathcal{B}^2}[2\mathcal{B}^{c}\mathcal{B}_{d[a}H^{d}{}_{b]c} - (\mathcal{D}^2\,\mathcal{B}\,\mathcal{B}^{d} + \mathcal{B}^{c}\mathcal{B}^{d}\,\mathcal{D}_c)H_{dab}] + + \frac{1}{\mathcal{B}^4}\,\mathcal{B}^{c}\mathcal{B}_{cd}\mathcal{B}^{d}\mathcal{B}^{e}H_{eab} = 0$$
$$\frac{1}{\alpha'}\bar{\beta}^{B} = \frac{1}{\alpha'}\frac{D-24}{48\pi^2} + \frac{1}{16\pi^2}[4\mathcal{B}^2 - K - \frac{1}{\mathcal{B}^2}(\mathcal{B}_{ab}\mathcal{B}^{ab} + \mathcal{B}^{a}\mathcal{B}^{b}K_{ab} + (\mathcal{D}^2\mathcal{B})^2) -$$
$$-\frac{1}{\mathcal{B}^4}(\frac{1}{4}G^{ab}\mathcal{B}^2)_{,a}(\mathcal{B}^2)_{,b} - \frac{1}{4}\mathcal{B}^{c}(\mathcal{B}^2)_{,c}\mathcal{D}^2\mathcal{B}) + \frac{1}{12}H_{abc}H^{abc}] = 0 \qquad (48)$$

The above equations are the conditions of the Weyl invariance in the theory (35) with the one-loop corrections. So we observe that the effective equations in the theory (35) with quantum gravity are much more complicated and have qualitatively different structure as compared with the ones that arise in the framework of the standard approach. Since the arbitrariness of the renormalization is restricted by the reparametrizations and conformal transformation of the metric, there is no any hope to reduce the equations (48) to standard form and therefore the difference between two sets of effective equations is essential. On the other hand, since we have used the harmonic gauge the starting model (35) is obviously renormalizable in it's original form and therefore the difference between ours and standard sets of effective equations can not be explained by the features of

the noncovariant conformal gauge. In this points our conclusions differ from the ones of Ref's [40, 41]. Despite the difference with standard approach was already discussed above let us tought this problem again.

The starting action of the theory with quantum gravity is not invariant under the Weyl transformation, and the two-dimensional metric in our theory has the nontrivial dynamics already at the classical level. Since the two-dimensional metric is quantized, the α' does not play the role of the loop expansion parameter. Whereas the standard one-loop conditions of Weyl invariance correspond to the first order in α', our one-loop conditions of Weyl invariance correspond to the first order in the expansion (40) which essentially includes the metric part. Thus our one-loop approximation does not correspond to the one-loop approximation in the standard approach. In fact the difference can be seen already at the tree level, since we are not able to extract the Weyl invariant classical part from the starting action (35) that is an important component of the standard approach. At the same time the account of loop corrections leads to the effective equations which strongly differ from the standard ones. In particular, (48) contains some higher derivative terms already at the one loop level, whereas in the standard approach these terms appear only at second loop.

The difference between our approach and the standard one is that we do not follow the order of integrations. Since we perform the integration over the metric simultaneously with the integration over the sigma model coordinates, our loop expansion of the effective action does not correspond to the expansion in the powers of α' and that is why the resulting effective equations are different at any finite loop. One can suppose that the difference between two approaches disappear if we sum up the contributions of all the higher loops and compare the full effective actions which arise in two approaches. Unfortunately at the moment this is very far from the real possibilities.

Conclusion

We have considered some aspects of the perturbative approach to the two-dimensional quantum gravity. In the covariant gauge the dilaton gravity (4) is quite similar to the dilaton models in $d = 4$ (see, for example, [49]). The main difference is, of course, the power counting renormalizability of gravity in $d = 2$. Moreover in $d = 2$ there is an additional identity for the quantum metric (14) that leads to the special features of quantum corrections like the strong dependence on the gauge fixing parameters. The use of the covariant gauge quarantees the renormalizability of the theory. That enables us to study the theory of string coupled to $2d$ gravity and calculate an alternative effective equations for the background fields.

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